Scaling laws for DOUAR

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1 Introduction

DOUAR works in a unit cube (linear dimension $L' = 1$). Any problem needs to be scaled from the natural world (linear dimension L). This spatial scaling must also be accompanied by other scalings, such as time, velocity, viscosity, cohesion, density, thermal diffusivity, etc.

Below are some of the most common scalings that may be required for using DOUAR. In each case, the scalings require that certain ratios remain equal in the real world and in DOUAR, and the relevant math will be shown, complete with the model parameters that can be set to one to simplify the scaling. Note that scaled values for DOUAR are denoted with primes.

2 Kinematically driven flow

The kinematic scaling should be used any time that nonzero velocity boundary conditions are applied. The number of required scalings varies depending on whether or not buoyancy forces are included in the calculation (see below).

2.1 No buoyancy

Without buoyancy forces, only time scaling is required because the flow pattern is independent of the value of the viscosity. The model time can be scaled as follows,

$$t' \frac{{v_0}'}{L'} = t \frac{v_0}{L} \tag{1}$$

where t is real world time in [s], v_0 is the prescribed reference velocity in [m/s], L is the reference length scale in [m], and t', v_0' and L' are the scaled model values. The model length scale (L') and velocity (${v_0}'$) should be set equal to one so that equation (1) simplifies to

$$t' = t \frac{v_0}{L} \frac{L'}{{v_0}'} = t \frac{v_0}{L}. \tag{2}$$

2.2 With buoyancy

For a kinematically driven flow with buoyancy, the balance between viscous (σ_v) and buoyancy (σ_g) forces must be conserved; thus the following dimensionless ratio must be identical for both the natural system and the scaled experiment,

$$\frac{\sigma_v}{\sigma_g} = \frac{\mu v_0/L}{(\rho g)L} = \frac{\mu v_0}{(\rho g)L^2} = \frac{\mu' v_0'}{(\rho g)' L'^2} \tag{3}$$

where μ is the viscosity in [Pa s], ρ is density in [kg/m^3] and g is the acceleration due to gravity in [m/s^2].

Thus, there are two simple scaling options: One can choose to set the model reference density/gravity, $(\rho g)'$, or the reference viscosity, μ', term equal to 1. In both cases, the reference velocity, v_0', is set to 1. If one chooses to scale the values by setting the reference $(\rho g)'$=1, then equation (3) can be used to calculate the scaled viscosity (μ'), as follows

$$\mu' = \frac{\mu v_0}{(\rho g)L^2} \frac{(\rho g)' L'^2}{v_0'} = \frac{\mu v_0}{(\rho g)L^2}. \tag{4}$$

Note that if you intend to use a model with multiple materials with different densities, you should always utilize the reference density value (i.e., that which is set equal to 1), not the density for the given material, when calculating the scaled material viscosity, μ'. If, instead, the reference viscosity, μ', is set equal to 1, equation (3) becomes

$$(\rho g)' = \frac{(\rho g)L^2}{\mu v_0} \frac{\mu' v_0'}{L'^2} = \frac{\rho g L^2}{\mu v_0}. \tag{5}$$

Again, be sure that the real world viscosity for the material with a scaled viscosity of 1 is used when scaling the density/gravity term in equation (5).

2.3 Thermal diffusivity

The balance between heat advection and conduction must be preserved. Thus the dimensionless Peclet number, Pe, must have the same value in both nature and the scaled experiment,

$$\mathrm{Pe} = \frac{L^2/\kappa}{L/v_0} = \frac{v_0 L}{\kappa} = \frac{v_0' L'}{\kappa'} \tag{6}$$

where κ is the rock thermal diffusivity in [m^2/s]. Rock thermal diffusivity is not always available in geologic literature, but can be calculated from several other rock thermal properties,

$$\kappa = \frac{k}{\rho c_p} \tag{7}$$

where k is the material thermal conductivity in [W/m K] and c_p is the material heat capacity in [J/kg K]. This means that the Peclet number can also be calculated as follows.

$$\mathrm{Pe} = \frac{v_0 L}{\kappa} = v_0 L \frac{\rho c_p}{k} \tag{8}$$

Thus the scaled diffusivity, κ', is given by setting the model velocity and length scales to 1.

$$\kappa' = \frac{\kappa}{v_0 L} {v_0}' L' = \frac{\kappa}{v_0 L} = \frac{1}{v_0 L} \frac{k}{\rho c_p} \tag{9}$$

Note that in this case, the material density, ρ, should be that of the material for which the thermal diffusivity is being scaled, which may or may not be the reference density value.

2.4 Heat production

Heat sources, such as radiogenic heat production in rocks, also need to be scaled for use in DOUAR. Geologic literature often contains volumetric heat production values reported in [μW/m^3], which will first need to be converted to [°C/My],

$$A = \frac{3.15576 \times 10^7 A_{vol}}{\rho c_p} \tag{10}$$

where 3.15576×10^7 is the number of seconds in a year and A_{vol} is the volumetric heat production in [μW/m^3]. The total amount of additional heat in the model from heat production needs to be scaled only by the model time and temperature,

$$\begin{aligned} A' &= 3.15576 \times 10^{-13} \frac{AL}{T_{max} v_0} \frac{{T_{max}}' {v_0}'}{L'} = 3.15576 \times 10^{-13} \frac{AL}{T_{max} v_0} \\ &= 1 \times 10^{-6} \frac{A_{vol}}{\rho c_p} \frac{L}{T_{max} v_0} \end{aligned} \tag{11}$$

where 3.15576×10^{-13} is one over the number of seconds in 1 My.

2.5 Cohesion or yield strength

When the problem involves a plastic (brittle) behavior, several parameters have the dimension of a stress, such as the cohesion, C_0, in the case of a frictional plastic behavior, or the yield strength, σ_0, in the case of the von Mises criterion. They must be scaled by the gravitational stress such that the following ratios are identical in both the natural and scaled experiments:

$$\frac{C_0}{(\rho g) L} = \frac{C_0'}{(\rho g)' L'}, \quad \frac{\sigma_0}{(\rho g) L} = \frac{\sigma_0'}{(\rho g)' L'} \tag{12}$$

Consequently the scaled cohesion and yield strength are given by:

$$C_0{}' = \frac{C_0}{(\rho g)L}(\rho g)' L' = \frac{C_0}{(\rho g)L}, \quad \sigma_0{}' = \frac{\sigma_0}{(\rho g)L}(\rho g)' L' = \frac{\sigma_0}{\rho g L} \tag{13}$$

Note that in cases where buoyancy forces are not included, these constants must be scaled by the viscous stresses, which gives:

$$C_0{}' = \frac{C_0 L}{\mu v_0}\frac{\mu' v_0{}'}{L'} = \frac{C_0 L}{\mu v_0}\mu', \quad \sigma_0{}' = \frac{\sigma_0 L}{\mu v_0}\frac{\mu' v_0{}'}{L'} = \frac{\sigma_0 L}{\mu v_0}\mu' \tag{14}$$

As above, you should always utilize the reference density value (i.e., that which is set equal to 1), not the density for the given material for this scaling. Non-dimensional constants such as the coefficient or angle of friction do not need to be scaled.

2.6 Non-linear viscosity

Rock viscosity is commonly represented by the following parameterization,

$$\mu = B\dot{\varepsilon}^{(1/n)-1} e^{Q/nR\bar{T}}, \tag{15}$$

where $\dot{\varepsilon}$ is the strain rate, B is a constant (dimension Pa s^{-n}), n is a dimensionless power, Q is the activation energy, R is the Boltzmann gas constant and $\bar{T}$ is the absolute temperature (in K).

The scaled temperature, T', is assumed to vary between 0 and 1, so temperature scaling is done by assuming a set maximum temperature $T_{\max}$,

$$T' = T/T_{\max}, \tag{16}$$

such that the exponential, Arrhenius, term in the expression for the viscosity becomes:

$$e^{Q/nR\bar{T}} = e^{Q/nR(T' T_{\max} + 273.15)} \tag{17}$$

The balance between viscous and buoyancy stresses must be preserved. Thus the following ratio must be the same in nature and the scaled experiment,

$$\frac{B(v_0/L)^{(1/n)-1}(v_0/L)}{(\rho g)L} = \frac{B v_0{}^{(1/n)}}{(\rho g)L^{(1/n)+1}} = \frac{B' v_0'{}^{(1/n)}}{(\rho g)' L'^{(1/n)+1}} \tag{18}$$

In DOUAR, the viscosity is expressed as

$$\mu = \mu_0 \dot{\varepsilon}^{(1/n)-1} e^{Q/nR\bar{T}} \tag{19}$$

and the constant μ_0 must, consequently, be scaled as follows,

$$\mu_0 = \frac{B v_0{}^{(1/n)}}{(\rho g)L^{(1/n)+1}} \frac{(\rho g)' L'^{(1/n)+1}}{v_0{}'^{(1/n)}} = \frac{B v_0{}^{(1/n)}}{(\rho g)L^{(1/n)+1}} \tag{20}$$

Again, as above, you should utilize the reference density value (i.e., that which is set equal to 1), not the density for the given material for this scaling.

3 Dynamically (buoyancy) driven flows

When no velocity is imposed and the flow is driven by an internal density difference, $\Delta\rho$, of dimension a, the only scaling that needs to be preserved is the aspect ratio of the problem:

$$a/L \tag{21}$$

If the flow has a free surface, the reduced density must also be preserved:

$$\Delta\rho/\rho \tag{22}$$

where ρ is the fluid reference density. The flow pattern is independent of the viscosity.

The characteristic time scale is obtained from the balance between viscous and buoyancy stresses,

$$t_{\mathrm{c}} = \frac{\mu}{(\rho g)L} \tag{23}$$

which must be equal in the dimensional and scaled experiments

$$t\frac{(\rho g)L}{\mu} = t'\frac{(\rho g)'L'}{\mu'}. \tag{24}$$

Thus, the time scaling is given by

$$t' = t\frac{(\rho g)L}{\mu}\frac{\mu'}{(\rho g)'L'} = t\frac{(\rho g)L}{\mu} \tag{25}$$

if the fluid buoyancy, (ρg), and viscosity, μ, have been set to one in the scaled experiment.